Precalculus B

Homework 6.2 Question 19

The exercise is this:

Find the vector(s) $\mathbf{v}$ satisfying the given conditions.

$\mathbf{u} = \langle 3,2 \rangle$, $\mathbf{u} \cdot \mathbf{v} = -8$, $|\mathbf{v}|^2 = 61$

From Section 6.2 Page 467 of the textbook we reference these definitions:

DEFINITION Dot Product

The **dot product** or **inner product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
3. $\mathbf{0} \cdot \mathbf{u} = 0$
4. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
   $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
5. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

Step 1: Use the information given to set up a system of equations

We know $u \bullet v = u_1 v_1 + u_2 v_2$ and from the given information $u \bullet v = -8$. We can set the two expressions equal to each other. Also replace the variables u_1 and u_2 with their actual values.

$$u_1 v_1 + u_2 v_2 = -8$$

$$3v_1 + 2v_2 = -8$$

We know $|v|^2 = v \bullet v = v_1 v_1 + v_2 v_2$ and from the given information $|v|^2 = 61$. We can set these two expressions equal to each other and rewrite the terms with exponents.

$$v_1 v_1 + v_2 v_2 = 61$$

$$v_1^2 + v_2^2 = 61$$

Now we have this system of two equations with two variables. We can solve the system.

$$3v_1 + 2v_2 = -8$$

$$v_1^{\ 2} + v_2^{\ 2} = 61$$

Step 2: Solve the system using Substitution method.

We learned in second year algebra how to solve systems using substitution. Typical problems used variables x and y instead. So let's temporarily use variables x and y in place of v_1 and v_2.

Let x = v_1

Let y = v_2

This gives us a system with x and y variables.

$$3x + 2y = -8$$

$$x^2 + y^2 = 61$$

Take the first equation and solve for y in terms of x.

$$\begin{aligned} 3x + 2y &= -8 \\ 2y &= -3x - 8 \\ y &= \frac{-3x-8}{2} \end{aligned}$$

Substitute this expression for y into the second equation. Multiply every term on both sides by 4. This will get rid of any fractions for now.

$$\begin{aligned} x^2 + y^2 &= 61 \\ x^2 + \left(\frac{-3x-8}{2}\right)^2 &= 61 \\ (4)x^2 + (4)\left(\frac{-3x-8}{2}\right)^2 &= (4)61 \\ 4x^2 + (-3x-8)^2 &= 244 \end{aligned}$$

Expand the binomial, combine like terms, and subtract 244 from both sides. Now we're ready to solve the quadratic using factoring, or else just use the Quadratic Formula.

$$\begin{aligned} &4x^2 + (-3x-8)^2 = 244 \\ &4x^2 + (9x^2 + 48x + 64) = 244 \\ &13x^2 + 48x + 64 = 244 \\ &13x^2 + 48x - 180 = 0 \end{aligned}$$

When we solve the quadratic we get two solutions:

$$x = 6$$

$$x = \frac{30}{13}$$

We substitute both solutions back into one of the earlier equations. That way we can obtain their corresponding y values. It's most convenient to use the equation where we already solved for y in terms of x. This was $y = \frac{-3x-8}{2}$.

For $x = 6$:

$$y = \frac{-3x-8}{2}$$

$$y = \frac{-3(6)-8}{2}$$

$$y = 5$$

For $x = \frac{30}{13}$:

$$y = \frac{-3\left(\frac{30}{13}\right)-8}{2}$$

$$y = \left(-\frac{90}{13}-8\right)\bullet\frac{1}{2}$$

$$y = \left(-\frac{90}{13}-\frac{104}{13}\right)\bullet\frac{1}{2}$$

$$y = \left(-\frac{194}{13}\right)\bullet\frac{1}{2}$$

$$y = -\frac{97}{13}$$

Step 3: Rewrite in vector form

We have two solutions that we solved using x and y variables. But we need to rewrite each of them as vector v.

$$(x, y) = (6, 5)$$

$$(x, y) = \left(\frac{30}{13}, -\frac{97}{13}\right)$$

$$v = \langle 6, 5 \rangle$$

$$v = \left\langle \frac{30}{13}, -\frac{97}{13} \right\rangle$$